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Abstract. The net absorption coefficient for multiphoton inverse bremsstrahlung is evaluated in closed form in the limit $\hbar\omega \ll kT$ over the full range of photon flux intensity. Weak- and strong-field expansions are presented and numerical results displayed.

1. Introduction

In laser-driven fusion experiments, the incident photon energy flux can be sufficiently high that nonlinear electromagnetic effects need be considered. The most direct of these is the multiphoton extension of the inverse bremsstrahlung process. Its explicit evaluation is a necessary first step to an understanding of laser absorption, providing the reference line from which to assess the possible impact of other, less well understood mechanisms. Inverse bremsstrahlung involves three characteristic energies: the photon energy $\hbar\omega$, the electron temperature kT and the 'quivering energy' $E_q = \frac{1}{2}m(eE/m\omega)^2$ with the mean-square electric field intensity $E^2 = 4\pi I/c$ where I is the irradiance; it is convenient to use the two dimensionless ratios $x = E_q/2kT$ and $y = \hbar\omega/2kT$. In laser fusion experiments, y is very small after the initial stage of plasma formation (typically, $\hbar\omega$ is about one eV for a Nd-glass laser while kT is several keV). This initial stage will not be discussed further here as it involves altogether different phenomenology in that the ionization is low, the counterpart atomic problem is quite different and more complicated (see the review article by Lambropoulos and Lambropoulos 1975) and other mechanisms such as impact ionization and charge exchange are important. The purpose of this paper is to calculate the net absorption coefficient for multiphoton inverse bremsstrahlung (more exactly, its ratio to the weak-field limit) for small y and the full range of x.

Multiphoton inverse bremsstrahlung and cognate matters have been a popular subject (Rand 1964, Silin 1965, Bunkin and Fedorov 1966, Hughes and Nicholson-Florence 1968, Brehme 1971, Nicholson-Florence 1971, Pert 1972, Osborn 1972, Seely and Harris 1973, Kroll and Watson 1973, Geltman and Teague 1974) but most of the interest has been in establishing the formalism at varying levels of sophistication, classically or quantum mechanically, and explicit results for the absorption coefficient are relatively meagre. For weak fields, apart from the verification of the single-photon limit, there is only a first-order correction for small y (Osborn 1972). (Rand (1964) and Bunkin and Fedorov (1966) quote first-order corrections in various limits to the cross section for an electron of given energy and direction.) For strong fields, some asymptotic results are given. Hughes and Nicholson-Florence (1968) obtain semi-quantitatively from Rand's treatment both the large- and small-y limits. They also derive an alternative strong-field model which arrives at a different answer but which has subsequently been amended (Nicholson-Florence 1971, Pert 1972), reconciling the conflict. Pert (1972) independently verifies the Rand–Hughes result and gives a qualitative discussion of the variation of the Coulomb collision logarithm with various parameters. A calculation of the effect of a high-frequency field on the plasma conductivity by Silin (1965) yields an effective collision frequency which is directly pertinent to the bremsstrahlung problem. He quotes a large-x and (effectively) small-y limit. All of the above asymptotic results behave as $x^{-3/2} \ln x$ with minor variations in the numerical coefficient and in the form of the collision logarithm (consistent within the level of rigour) for small y and not very differently for large y. On the other hand, the heuristic evaluation of the asymptotic limit for small y of Osborn (1972) lacks the $\ln x$ factor, as does the Seely and Harris (1973) result for large y which further omits the collision logarithm. No explicit intermediate-field expressions exist, though Silin (1965) quotes an integral representation for which he tabulates some numerical values. (Nicholson-Florence (1971) integrates the Bunkin and Fedorov (1966) cross sections numerically to obtain the n-photon net absorption coefficients (for $n = 1$–4) for an electron of given energy over what amounts to a considerable range in x and y.)

2. Evaluation of multiphoton correction factor to inverse bremsstrahlung absorption coefficient

Osborn (1972) transforms the usual momentum-space integral to a double infinite sum involving modified Bessel functions. Rewritten in terms of x and y and corrected for a typographical error, his equation (23) reads

$$F = \frac{3}{\sinh yK_0(y)} \sum_{n=1}^{\infty} n \sinh ny \sum_{k=0}^{\infty} \frac{(-1)^k(2n+2k)!}{(2n+2k+1)(2n+k)![(n+k)!]^2 k!} \times \left(\frac{nx}{2y}\right)^{n+k-1} K_{n+k-1}(ny) \tag{1}$$

with F the ratio of the multiphoton absorption coefficient to its weak-field (single-photon) value. Using $j = n+k-1$ as index instead of k and interchanging the order of summation, equation (1) becomes

$$F = \frac{3}{\sinh yK_0(y)} \sum_{j=0}^{\infty} \frac{(x/2y)^j}{(2j+3)[(j+1)!]^2} \sum_{n=1}^{j+1} \frac{(-1)^{j+1-n}(2j+2)!}{(j+1+n)!(j+1-n)!} n^{j+1} \sinh nyK_j(ny). \tag{2}$$

The $j = 0$ term is, of course, 1. The modified Bessel functions can be expanded as (Abramowitz and Stegun 1964)

$$K_j(ny) = \frac{1}{2}\left(\frac{ny}{2}\right)^{-j} \sum_{k=0}^{j-1} \frac{(j-1-k)!}{k!}\left(-\frac{n^2y^2}{4}\right)^k + (-1)^{j+1} \ln\left(\frac{ny}{2}\right) I_j(ny) + \frac{1}{2}(-1)^j\left(\frac{ny}{2}\right)^j \sum_{k=0}^{\infty} (\psi(k+1)+\psi(j+k+1)) \frac{(n^2y^2/4)^k}{k!(j+k)!} \tag{3}$$

for $j > 0$ (and also for $j = 0$ with the first sum omitted). In the small-y limit, the last sum is clearly of order y^j whereas the middle term is of order $y^j \ln y$, since

$$I_j(ny) \simeq (ny/2)^j/j!, \tag{4}$$

and hence larger. The first sum is dominant in equation (3), but its leading contribution vanishes in the sum over n: expanding sinh ny in odd powers $(2l+1)$ of ny, and abbreviating $2i \equiv 2l+2k+2$, the n summation is

$$\sum_{n=1}^{j+1}(-1)^{j+1-n}\frac{(2j+2)!n^{2i}}{(j+1+n)!(j+1-n)!}=\frac{1}{2}\sum_{m=0}^{2j+2}(-1)^m\frac{(2j+2)!(j+1-m)^{2i}}{m!(2j+2-m)!}. \tag{5}$$

This sum has been evaluated in a statistical theorem (Feller 1957) as

$$\sum_{m=0}^{n}(-1)^m m^k n!/m!(n-m)! = \delta_{kn}n!, \qquad k \leqslant n, \tag{6}$$

so that equation (5) vanishes if $i < j+1$. The remaining contribution from the first sum in equation (3) is thus comparable to that from the last sum. Upon retaining only the dominant term in ln y, using equation (4), and performing the n sum as in equations (5) and (6), F reduces to

$$F=\frac{3}{2}\sum_{j=0}^{\infty}\left(-\frac{x}{4}\right)^j\frac{(2j+2)!}{(2j+3)j![(j+1)!]^2}. \tag{7}$$

On applying the duplication formula for Γ functions, equation (7) can be identified as the generalized hypergeometric series

$$F = {}_2F_2(\tfrac{3}{2},\tfrac{3}{2};\tfrac{5}{2},2;-x) \tag{8}$$

which converges absolutely for all x.

Equation (8) is the small-y limit of the solution to equation (1) for arbitrary finite x. That does not ensure, however, that, for small but finite y, an unrestricted large-x limit applied to equation (8) will yield the corresponding solution to equation (1), as the $F(x)$ of equation (8) decreases with x and the higher-order terms in y might conceivably overtake it. This issue has not been resolved in principle because the asymptotic limit in x for fixed y of equation (1) has not been obtained, nor indeed even the asymptotic limit of the next order term in y. The latter term (of the form $-f(x)/\ln y$) has, however, been evaluated numerically for x up to 15 (beyond which the round-off error is overwhelming). The value of $f(x)$ rises from 0 at $x=0$ to 0·28 at $x=3$ and then declines, albeit a bit more slowly with x than $F(x)$. At $x=15$, $f(x)\simeq 1{\cdot}7\,F(x)$. For a Nd-glass laser and $kT \simeq 3$ keV, $x=15$ corresponds to an irradiance of about 10^{18} W cm^{-2}, a couple of orders of magnitude above current peak values.

3. Integral representation of F and asymptotic value

A complementary approach is to use judicious differentiation of equation (7) to obtain

$$\tfrac{2}{3}x^{-1/2}\frac{\mathrm{d}}{\mathrm{d}x}(x^{3/2}F) = {}_1F_1(\tfrac{3}{2};2;-x) = \mathrm{e}^{-x/2}(I_0(x/2)-I_1(x/2)) \tag{9}$$

from the properties of the confluent hypergeometric functions and the modified Bessel functions (Abramowitz and Stegun 1964). The integral representation

$$F = \tfrac{3}{2}x^{-3/2}\int_0^x y^{1/2}\,\mathrm{d}y\,\exp(-y/2)(I_0(y/2)-I_1(y/2)) \tag{10}$$

follows immediately. This agrees precisely with the Silin (1965) expression, a remarkable result in view of the lack of resemblance between the formulations and solutions. The putative validity constraints of the two derivations do not coincide but appear to be compatible. Silin's conditions are

$$\omega \ll v_t/b_{\max} < v_t/b_{\min} \tag{11}$$

with v_t the thermal velocity and the b's limiting impact parameters in the collision logarithm expressed as $\ln(b_{\max}/b_{\min})$. In the range of plasmas of interest for laser-driven fusion, $b_{\min}$ is the de Broglie wavelength, $b_{\max}$ the ion–electron Debye length (Brysk *et al* 1975). The outer inequality in equation (11) thus reduces to $\hbar\omega \ll kT$ or $y \ll 1$, in agreement with the present work. The meaning of the left-side inequality in equation (11) is less clear (if not counter-intuitive); it can be relaxed in Silin's derivation of equation (10). The issue is not really relevant in the present context, as the collision logarithm of the conductivity calculation is supplanted by a Gaunt factor for bremsstrahlung (which is a different function).

The behaviour of F for large x is not easily discerned from equation (10), and numerical integration over a wide range of values of x is rather tedious. It is both more instructive and more convenient to resort to the asymptotic representation of F. The ${}_2F_2$ of equation (8) is a degenerate form not reducible to a Meijer G function, hence not susceptible to the standard asymptotic techniques (and expected to include a logarithmic term). The evaluation is achieved with the artifice of inserting a factor $y^{-\epsilon}$ into the integrand of equation (10) and reverting to the confluent hypergeometric function of equation (9). The definite integral is then known (Magnus *et al* 1966),

$$\int_0^\infty y^{1/2-\epsilon}\,\mathrm{d}y\,{}_1F_1(\tfrac{3}{2};2;-y) = \Gamma(\tfrac{3}{2}-\epsilon)\Gamma(\epsilon)/\Gamma(\tfrac{3}{2})\Gamma(\tfrac{1}{2}+\epsilon). \tag{12}$$

In the residual integral, the asymptotic expansion (Dingle 1973)

$${}_1F_1(\tfrac{3}{2};2;-y) \sim 2(\pi y)^{-3/2}\sum_{j=0}^{\infty}\Gamma(j+\tfrac{1}{2})\Gamma(j+\tfrac{3}{2})/j!y^j \tag{13}$$

can be integrated to yield

$$\int_x^\infty y^{1/2-\epsilon}\,\mathrm{d}y\,{}_1F_1(\tfrac{3}{2};2;-y) = 2\pi^{-3/2}\sum_{j=0}^{\infty}\Gamma(j+\tfrac{1}{2})\Gamma(j+\tfrac{3}{2})/(j+\epsilon)j!x^{j+\epsilon}. \tag{14}$$

For small ϵ, equation (12) can be reduced by

$$\Gamma(\epsilon) = \epsilon^{-1}\Gamma(1+\epsilon), \qquad \Gamma(a+\epsilon) = \Gamma(a)(1+\epsilon\psi(a)+\ldots), \tag{15}$$

whereas in equation (14)

$$x^{-\epsilon} = \exp(-\epsilon\ln x) = 1-\epsilon\ln x+\ldots. \tag{16}$$

The ϵ^{-1} terms cancel. On dropping terms of order ϵ, there remains

$$F \sim \tfrac{3}{2}\pi^{-1/2}x^{-3/2}\left(\ln x+\gamma+4\ln 2-2-2\pi^{-1}\sum_{j=1}^{\infty}\Gamma(j+\tfrac{1}{2})\Gamma(j+\tfrac{3}{2})/jj!x^j\right). \tag{17}$$

The leading (logarithmic) term agrees with the asymptotic limit of Silin (1965), of course; in fact, his numerical estimate of the next (constant) term is rather good.

4. Numerical results

F was evaluated numerically by summing the series of equation (8) for small to moderate x and the series of equation (17) for large x (overlapping for $x = 10$–30 to verify consistency). In figure 1 F is displayed as a function of x.

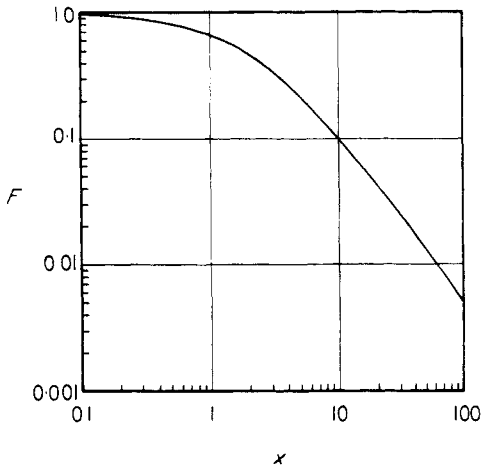

Figure 1. Multiphoton correction factor, F, to inverse bremsstrahlung absorption coefficient as a function of x ($= E_q/2kT$).

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